Exact Periodic Wave Solutions to the Melnikov Equation

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Z. Naturforsch. **60a,** 321 – 327 (2005); received December 29, 2004

Exact periodic wave solutions in terms of the Jacobi elliptic functions are obtained to the Melnikov equation by means of the extended mapping method with symbolic computation. The stability of these periodic waves is numerically studied. The results show that the linearly combined waves of cn− and dn− functions can propagate stably. For the blow up solutions, the linearly combined periodic solutions of nd− and sd− functions and of dc− and sc− functions can also propagate stably and others can not do. Under the limit conditions, some new solitary wave solutions and trigonometric periodic wave solutions are got. The method is applicable to a large variety of nonlinear partial differential equations. – PACS: 05.45.Yv, 02.30.Ik, 02.30.Jr

Key words: The Melnikov Equation; Exact Solutions; Extended Mapping Method.

1. Introduction

It is interesting that exact solutions to some nonlinear partial differential equations (PDEs) may be expressed by the Jacobi elliptic functions [1 – 10]. The Jacobi elliptic functions $\mathrm{sn}\xi = \mathrm{sn}(\xi|m)$, $\mathrm{cn}\xi = \mathrm{cn}(\xi|m)$, and $\mathrm{dn}\xi = \mathrm{dn}(\xi|m)$, where $m(0 < m < 1)$ is the modulus of the elliptic function, are double periodic and possess properties of triangular functions, namely, $\mathrm{sn}^2\xi + \mathrm{cn}^2\xi = 1, \mathrm{dn}^2\xi + m^2\mathrm{sn}^2\xi = 1$, $(\mathrm{sn}\xi)' = \mathrm{cn}\xi\,\mathrm{dn}\xi$, $(\mathrm{cn}\xi)' = -\mathrm{sn}\xi\,\mathrm{dn}\xi$, $(\mathrm{dn}\xi)' = -m^2\mathrm{sn}\xi\,\mathrm{cn}\xi$. When $m \to 0$, the Jacobi elliptic functions degenerate to the trigonometric functions, i. e. $\mathrm{sn}\xi \to \sin\xi$, $\mathrm{cn}\xi \to \cos\xi$, $\mathrm{dn}\xi \to 1$. When $m \to 1$, the Jacobi elliptic functions degenerate to the hyperbolic functions, i. e. $\mathrm{sn}\xi \to \tanh\xi$, $\mathrm{cn}\xi \to \mathrm{sech}\xi$, $\mathrm{dn}\xi \to \mathrm{sech}\xi$. Detailed explanations about Jacobi elliptic functions can be found in [7 – 10]. Although the other 9 Jacobi elliptic functions are all expressible in terms of $\mathrm{sn}\xi$, $\mathrm{cn}\xi$ and $\mathrm{dn}\xi$, we should allow m to be outside the range $0 < m < 1$ [7]. In the earlier papers, we obtained the exact periodic wave solutions in terms of the Jacobi elliptic functions for some nonlinear PDEs by using the mapping method [5 – 6]. In this paper, we will use the extended mapping method with symbolic computation to study exact travelling wave solutions to nonlinear PDEs.

2. Results and Discussion

We take the Melnikov equation [11]

$$iu_t + u_{xx} + uv = 0, \quad v_t + v_y + (|u|^2)_x = 0, \tag{1}$$

as a simple illustrative example, the integrability of which has been recently discussed by Porsezian [12]. However, (1) possesses many interesting solution structures which have not yet been found. Substituting $u = \phi(\xi)e^{i\eta}$, $\quad v = \psi(\xi)$, $\quad \xi = kx + ly - \omega t$, and $\eta = Kx + Ly - \Omega t$ into (1) and setting $\omega = 2Kk$, we obtain the equation for ϕ and ψ

$$\begin{aligned} &k^2\phi'' + (\Omega - K^2)\phi + \phi\psi = 0, \\ &(l-\omega)\psi + k\phi^2 = C_1, \end{aligned} \tag{2}$$

where the second of (1) has been integrated once and C_1 is the constant of integration. The substitution of the last of (2) into the first one yields

$$A\phi'' + B\phi + C\phi^3 = 0, \tag{3}$$

with

$$\begin{aligned} &A = k^2(l-\omega),\ B = C_1 + (l-\omega)(\Omega - K^2), \\ &C = -k. \end{aligned} \tag{4}$$

Now we assume that (3) has the solution of the form

$$\phi(\xi) = A_0 + A_1 f + B_1 g, \tag{5}$$

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where A_i and B_1 are constants to be determined, and f and g satisfy the system of equations

$$\begin{aligned} &f'' = pf + qf^3, \quad f'^2 = pf^2 + \frac{1}{2}qf^4 + r, \\ &g'' = g(c_1 + c_2 f^2), \quad g^2 = c_3 + c_4 f^2. \end{aligned} \tag{6}$$

Here the prime means the derivatives with respect to ξ, and p, q, r and c_i are constants to be determined. Thus a new algebraic mapping relation is established through (5) between the solution to (3) and that of (6).

Remark 1. If we take $B_1 = 0$, this is the mapping method [5–6]. Notice that if we use the modified mapping method [13–14], g in (5) is replaced by f^{-1}.

Remark 2. Due to the entrance of the parameters p, q, r and c_i, (6) has rich structures of solutions. As $p = -2, q = 2, r = 1$ and $c_1 = -1, c_2 = 2, c_3 = 1, c_4 = -1$, for example, the solution of (6) reads $f(\xi) = \tanh\xi$, $g(\xi) = \mathrm{sech}\xi$, and the method is called the two-family truncation method [15]. By choosing the different values of the parameters, as will be seen in the following, we may obtain abundant solutions to the PDE in question. Notice that the first two of (6) are discussed in detail in the mapping method [5, 6] while the rest is a generalization of the relation of $f = \tanh\xi$ and $g = \mathrm{sech}\xi$ and involves the computation of the function g and the constants c_1, c_2 and c_3, c_4.

Substituting (5) with (6) into (3) and equating the coefficients of like powers of $f^i g^j (j = 0, 1)$ to zero we get two sets of solutions (using MATHEMATICA)

$$\begin{aligned} &A_0 = 0, \quad A_1 = \pm\sqrt{-\frac{qA}{C}}, \\ &B_1 = 0, \quad pA + B = 0, \end{aligned} \tag{7}$$

and

$$\begin{aligned} &A_0 = 0, \quad A_1 = \pm\sqrt{\frac{(c_4 p - c_3 q)A + c_4 B}{c_3 C}}, \\ &B_1 = \pm\sqrt{-\frac{pA + B}{3c_3 C}}, \quad (3c_1 - p)A + 2B = 0, \\ &(3c_4 p - 3c_3 q - c_1 c_4 + c_2 c_3)A + 2c_4 B = 0. \end{aligned} \tag{8}$$

Because the first solution equation (7) can also be obtained by the mapping method, the discussion about it is only a routine thing following [5–6]. So we omit it here. From (8) and using (4) we get the exact solution of (1) as follows:

$$\begin{aligned} u &= [\pm\sqrt{\frac{(3c_1 c_4 - 3c_4 p + 2c_3 q)k(l - 2Kk)}{2c_3}} f(\xi) \\ &\pm\sqrt{\frac{(p - c_1)k(l - 2Kk)}{2c_3}} g(\xi)]\mathrm{e}^{\mathrm{i}\eta}, \\ v &= -\frac{k}{l - 2Kk}|u|^2 + \frac{C_1}{l - 2Kk}, \end{aligned} \tag{9}$$

where f and g satisfy (6) with the constraint among the parameters

$$4c_4 p - 3c_3 q - 4c_1 c_4 + c_2 c_3 = 0,$$

and $\xi = kx + ly - 2Kkt$ and $\eta = Kx + Ly - \Omega t$ with

$$\Omega = K^2 - \frac{1}{2}(3c_1 - p)k^2 - \frac{C_1}{l - 2Kk}.$$

Other parameters are arbitrary constants in the sense that the expressions in the square root should be positive and $l \neq 2Kk$ and $c_3 \neq 0$. It is worth noticing the

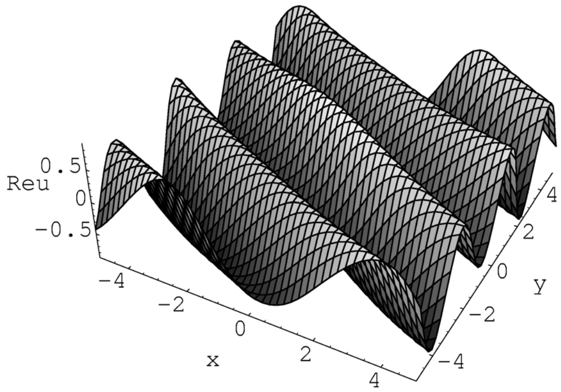

Fig. 1. The real part spatial structure of u for (10).

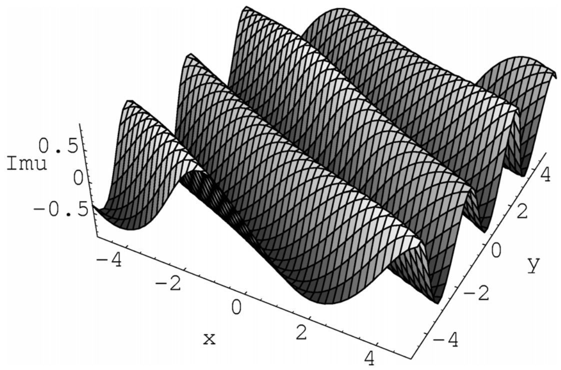

Fig. 2. The imaginary part spatial structure of u for (10).

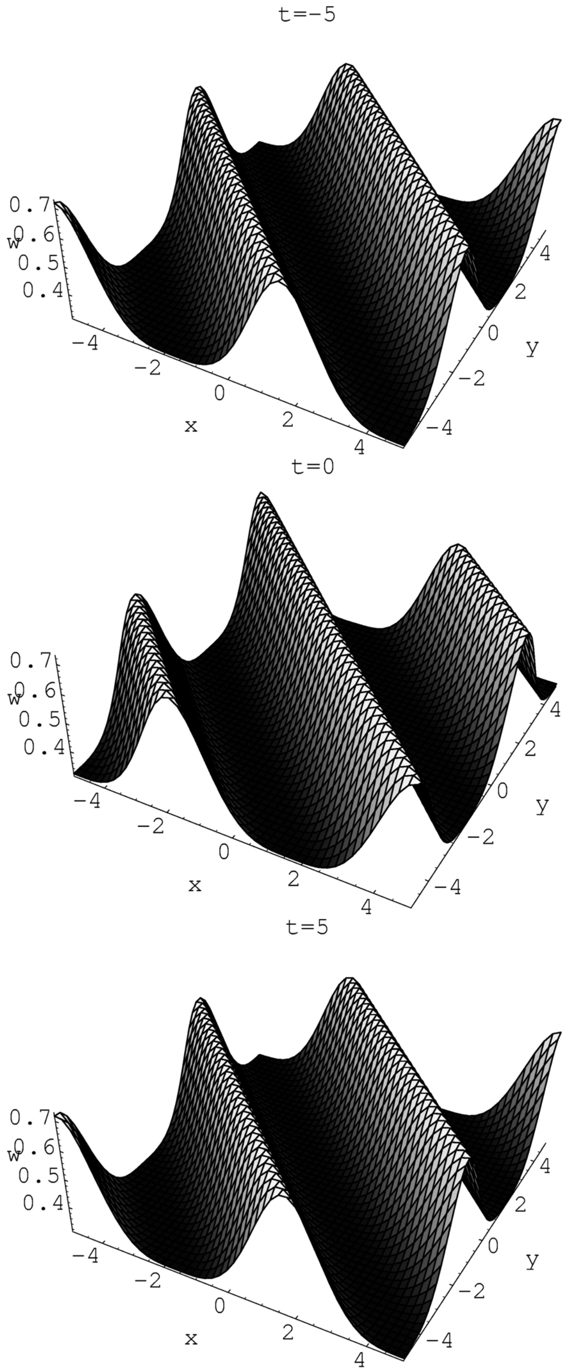

Fig. 3. The evolution graphs of $|u|^2$ for (10).

choice for the positive and negative signs in (9) is arbitrary. These settlements are valid throughout the paper. In what follows we discuss the specific expression of f and g according to (6) as examples.

Case 1. $p = 2m^2 - 1$, $q = -2m^2$, $r = 1 - m^2$, $c_1 = m^2$, $c_2 = -2m^2$, $c_3 = 1 - m^2$, $c_4 = m^2$.

In this case, we have $f(\xi) = \mathrm{cn}\xi$, $g(\xi) = \mathrm{dn}\xi$. Thus the periodic wave solution of (1) is

$$\begin{aligned} u &= \pm\frac{1}{2}\sqrt{2k(2Kk-l)}[m\mathrm{cn}(kx+ly-2Kkt) \\ &\quad + \mathrm{dn}(kx+ly-2Kkt)]\mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\ v &= \frac{1}{2}k^2[m\mathrm{cn}(kx+ly-2Kkt) \\ &\quad + \mathrm{dn}(kx+ly-2Kkt)]^2 + \frac{C_1}{l-2Kk}, \end{aligned} \tag{10}$$

where

$$\Omega = K^2 - \frac{1}{2}(1+m^2)k^2 - \frac{C_1}{l-2Kk}. \tag{11}$$

In order to understand intuitively the properties of the Jacobi elliptic wave solutions, we draw the plots of the real part, the imaginary part of u (taking the positive sign) (see Figs. 1, 2). The evolution graphs of $w \equiv |u|^2$ (Fig. 3) indicate that w can propagate stably and the parameters are $k = l = K = 1$, $L = 2$, $m = 0.2$, $C_1 = 0$ and $t = 0$. For the chosen parameters, the wavelength of these waves is $\sqrt{2}\pi$. As $m \to 1$, (10) degenerates to

$$\begin{aligned} u &= \pm\sqrt{2k(2Kk-l)}\mathrm{sech}(kx+ly-2Kkt) \\ &\quad \cdot \mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\ v &= 2k^2\mathrm{sech}^2(kx+ly-2Kkt) + \frac{C_1}{l-2Kk}, \end{aligned} \tag{12}$$

where

$$\Omega = K^2 - k^2 - \frac{C_1}{l-2Kk}, \tag{13}$$

which is the solitary wave solution of (1).

Case 2. $p = 2 - m^2$, $q = -2(1-m^2)$, $r = -1$, $c_1 = 1$, $c_2 = -2(1-m^2)$, $c_3 = -\frac{1}{m^2}$, $c_4 = \frac{1}{m^2}$.

The solution of (6) reads $f(\xi) = \mathrm{nd}\xi \equiv 1/\mathrm{dn}\xi$, $g(\xi) = \mathrm{sd}\xi \equiv \mathrm{sn}\xi/\mathrm{dn}\xi$. So we get the periodic wave solution of (1)

$$\begin{aligned} u &= \pm\frac{1}{2}\sqrt{2(1-m^2)k(2Kk-l)}[\mathrm{nd}(kx+ly-2Kkt) \\ &\quad + m\mathrm{sd}(kx+ly-2Kkt)]\mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\ v &= \frac{1}{2}(1-m^2)k^2[\mathrm{nd}(kx+ly-2Kkt) \\ &\quad + m\mathrm{sd}(kx+ly-2Kkt)]^2 + \frac{C_1}{l-2Kk}, \end{aligned} \tag{14}$$

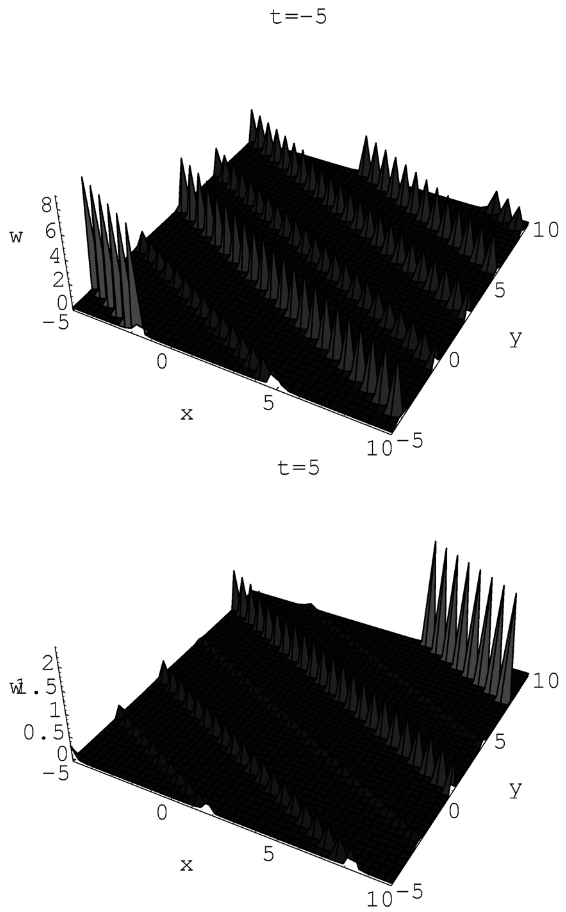

Fig. 4. The evolution graphs of $|u|^2 * 10^{-3}$ for (15).

with Ω given by (11). The evolution figures of $|u|^2$ for (14) are very similar with those for (10) and omitted, so $|u|^2$ of (14) can also propagate stably.

Case 3. $p = 2 - m^2$, $q = 2$, $r = 1 - m^2$, $c_1 = 1$, $c_2 = 2, c_3 = 1 - m^2$, $c_4 = 1$.

From (6) one obtains $f(\xi) = \mathrm{cs}\xi \equiv \mathrm{cn}\xi/\mathrm{sn}\xi$, $g(\xi) = \mathrm{ds}\xi \equiv \mathrm{dn}\xi/\mathrm{sn}\xi$. And the periodic wave solution of (1) reads

$$
\begin{aligned}
u &= \pm\frac{1}{2}\sqrt{2k(l-2Kk)}[\mathrm{cs}(kx+ly-2Kkt) \\
&\quad + \mathrm{ds}(kx+ly-2Kkt)]\mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\
v &= -\frac{1}{2}k^2[\mathrm{cs}(kx+ly-2Kkt) \\
&\quad + \mathrm{ds}(kx+ly-2Kkt)]^2 + \frac{C_1}{l-2Kk},
\end{aligned}
\tag{15}
$$

with Ω given by (11). From Fig. 4, one can easily see that $w \equiv |u|^2 * 10^{-3}$ of (15) can not propagate stably, where the parameters are $k = 1$, $l = 2$, $m = 0.2$, $K = 0.5$, which are valid for all the figures in the following. The wavelength is obviously $2\pi/\sqrt{5}$ for these parameters.

As $m \to 0$ and $m \to 1$, (15) degenerates to

$$
\begin{aligned}
u &= \pm\frac{1}{2}\sqrt{2k(l-2Kk)}[\cot(kx+ly-2Kkt) \\
&\quad + \csc(kx+ly-2Kkt)]\mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\
v &= -\frac{1}{2}k^2[\cot(kx+ly-2Kkt) \\
&\quad + \csc(kx+ly-2Kkt)]^2 + \frac{C_1}{l-2Kk},
\end{aligned}
\tag{16}
$$

with

$$
\Omega = K^2 - \frac{1}{2}k^2 - \frac{C_1}{l-2Kk},
\tag{17}
$$

and

$$
\begin{aligned}
u &= \pm\sqrt{2k(l-2Kk)}\,\mathrm{csch}(kx+ly-2Kkt) \\
&\quad \cdot \mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\
v &= -2k^2\mathrm{csch}^2(kx+ly-2Kkt),
\end{aligned}
\tag{18}
$$

with Ω given by (13), respectively.

Case 4. $p = 2m^2 - 1$, $q = 2(1-m^2)$, $r = -m^2$, $c_1 = m^2$, $c_2 = 2(1-m^2)$, $c_3 = -1$, $c_4 = 1$.

Equation (6) has the solution $f(\xi) = \mathrm{nc}\xi \equiv 1/\mathrm{cn}\xi$, $g(\xi) = \mathrm{sc}\xi \equiv \mathrm{sn}\xi/\mathrm{cn}\xi$. From (9) we get

$$
\begin{aligned}
u &= \pm\frac{1}{2}\sqrt{2(1-m^2)k(l-2Kk)}[\mathrm{nc}(kx+ly-2Kkt) \\
&\quad + \mathrm{sc}(kx+ly-2Kkt)]\mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\
v &= -\frac{1}{2}k^2(1-m^2)[\mathrm{nc}(kx+ly-2Kkt) \\
&\quad + \mathrm{sc}(kx+ly-2Kkt)]^2 + \frac{C_1}{l-2Kk},
\end{aligned}
\tag{19}
$$

with Ω given by (11). As $m \to 0$, from (19) we get the trigonometric periodic wave solution of (1)

$$
\begin{aligned}
u &= \pm\frac{1}{2}\sqrt{2k(l-2Kk)}[\sec(kx+ly-2Kkt) \\
&\quad + \tan(kx+ly-2Kkt)]\mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\
v &= -\frac{1}{2}k^2[\sec(kx+ly-2Kkt) \\
&\quad + \tan(kx+ly-2Kkt)]^2 + \frac{C_1}{l-2Kk},
\end{aligned}
\tag{20}
$$

with Ω given by (17).

Case 5. $p = -(1+m^2), \quad q = 2, \quad r = m^2.$

Subcase 5.1 $c_1 = -m^2,\ c_2 = 2,\ c_3 = -1,\ c_4 = 1.$ Equation (6) has the solution $f(\xi) = \mathrm{ns}\xi \equiv 1/\mathrm{sn}\xi$, $g(\xi) = \mathrm{cs}\xi \equiv \mathrm{cn}\xi/\mathrm{sn}\xi$. Thus we get the periodic wave solution of (1)

$$\begin{aligned} u &= \pm\frac{1}{2}\sqrt{2k(l-2Kk)}[\mathrm{ns}(kx+ly-2Kkt) \\ &\quad + \mathrm{cs}(kx+ly-2Kkt)]\mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\ v &= -\frac{1}{2}k^2[\mathrm{ns}(kx+ly-2Kkt) \\ &\quad + \mathrm{cs}(kx+ly-2Kkt)]^2 + \frac{C_1}{l-2Kk}, \end{aligned} \tag{21}$$

with

$$\Omega = K^2 - \frac{1}{2}(1-2m^2)k^2 - \frac{C_1}{l-2Kk}. \tag{22}$$

As $m \to 0$, (21) degenerates to (16). And as $m \to 1$, from (21) we obtain

$$\begin{aligned} u &= \pm\frac{1}{2}\sqrt{2k(l-2Kk)}[\coth(kx+ly-2Kkt) \\ &\quad + \mathrm{csch}(kx+ly-2Kkt)]\mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\ v &= -\frac{1}{2}k^2[\coth(kx+ly-2Kkt) \\ &\quad + \mathrm{csch}(kx+ly-2Kkt)]^2 + \frac{C_1}{l-2Kk}, \end{aligned} \tag{23}$$

with

$$\Omega = K^2 + \frac{1}{2}k^2 - \frac{C_1}{l-2Kk}. \tag{24}$$

Subcase 5.2 $c_1 = -1, c_2 = 2, c_3 = -m^2, c_4 = 1.$ The solution of (6) reads $f(\xi) = \mathrm{ns}\xi$, $g(\xi) = \mathrm{ds}\xi \equiv \mathrm{dn}\xi/\mathrm{sn}\xi$. So another periodic wave solution of (1) is

$$\begin{aligned} u &= \pm\frac{1}{2}\sqrt{2k(l-2Kk)}[\mathrm{ns}(kx+ly-2Kkt) \\ &\quad + \mathrm{ds}(kx+ly-2Kkt)]\mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\ v &= -\frac{1}{2}k^2[\mathrm{ns}(kx+ly-2Kkt) \\ &\quad + \mathrm{ds}(kx+ly-2Kkt)]^2 + \frac{C_1}{l-2Kk}, \end{aligned} \tag{25}$$

with

$$\Omega = K^2 - \frac{1}{2}(-2+m^2)k^2 - \frac{C_1}{l-2Kk}. \tag{26}$$

As $m \to 0$, (25) degenerates to

$$\begin{aligned} u &= \pm\sqrt{2k(l-2Kk)}\csc(kx+ly-2Kkt) \\ &\quad \cdot \mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\ v &= -2k^2\csc^2(kx+ly-2Kkt) + \frac{C_1}{l-2Kk}, \end{aligned} \tag{27}$$

with

$$\Omega = K^2 + k^2 - \frac{C_1}{l-2Kk}. \tag{28}$$

As $m \to 1$, from (25) we obtain (23) again.

Subcase 5.3 $c_1 = -1, \quad c_2 = 2, \quad c_3 = -\frac{m^2}{1-m^2}$, $c_4 = \frac{1}{1-m^2}$.
The solution of (6) is $f(\xi) = \mathrm{dc}\xi \equiv \mathrm{dn}\xi/\mathrm{cn}\xi$, $g(\xi) = \mathrm{nc}\xi \equiv 1/\mathrm{cn}\xi$. Therefore, we have

$$\begin{aligned} u &= \pm\frac{1}{2}\sqrt{2k(l-2Kk)}[\mathrm{dc}(kx+ly-2Kkt) \\ &\quad + \sqrt{1-m^2}\mathrm{nc}(kx+ly-2Kkt)]\mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\ v &= -\frac{1}{2}k^2[\mathrm{dc}(kx+ly-2Kkt) \\ &\quad + \sqrt{1-m^2}\mathrm{nc}(kx+ly-2Kkt)]^2 + \frac{C_1}{l-2Kk}, \end{aligned} \tag{29}$$

with Ω given by (26). As $m \to 0$, (29) degenerates to

$$\begin{aligned} u &= \pm\sqrt{2k(l-2Kk)}\sec(kx+ly-2Kkt) \\ &\quad \cdot \mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\ v &= -2k^2\sec^2(kx+ly-2Kkt) + \frac{C_1}{l-2Kk}, \end{aligned} \tag{30}$$

with Ω given by (28). Notice that the evolution figures of $|u|^2$ for case 4–5.3 are very similar with those for case 3 and omitted, so $|u|^2$ of these cases can also not propagate stably.

Subcase 5.4 $c_1 = -m^2, \quad c_2 = 2, \quad c_3 = -\frac{1}{1-m^2}$, $c_4 = \frac{1}{1-m^2}$.
One has $f(\xi) = \mathrm{dc}\xi$, $g(\xi) = \mathrm{sc}\xi \equiv \mathrm{sn}\xi/\mathrm{cn}\xi$. So the periodic wave solution of (1) reads

$$\begin{aligned} u &= \pm\frac{1}{2}\sqrt{2k(l-2Kk)}[\mathrm{dc}(kx+ly-2Kkt) \\ &\quad + \sqrt{1-m^2}\mathrm{sc}(kx+ly-2Kkt)]\mathrm{e}^{\mathrm{i}(Kx+Ly-\Omega t)}, \\ v &= -\frac{1}{2}k^2[\mathrm{dc}(kx+ly-2Kkt) \\ &\quad + \sqrt{1-m^2}\mathrm{sc}(kx+ly-2Kkt)]^2 + \frac{C_1}{l-2Kk}, \end{aligned} \tag{31}$$

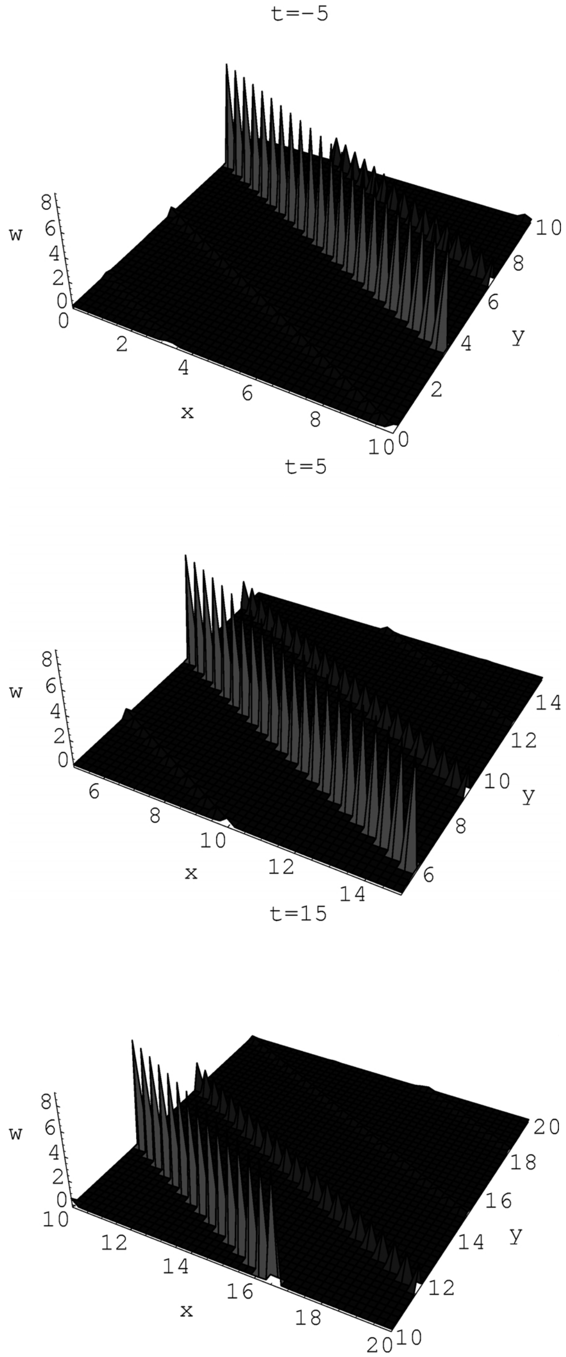

Fig. 5. The evolution graphs of $|u|^2 * 10^{-3}$ for (31).

with Ω given by (22). The evolution graphs (Fig. 5) of $w \equiv |u|^2 * 10^{-3}$ for (31) is surprising because we see that $w \equiv |u|^2 * 10^{-3}$ can propagate stably! As $m \to 0$, (31) degenerates to (20).

3. Conclusion

Exact travelling wave solutions to the Melnikov equation studied by means of the extended mapping method with symbolic computation. Abundant periodic wave solutions in terms of the Jacobi elliptic functions are obtained. Limit cases are studied and exact solitary wave solutions and trigonometric periodic wave solutions are got. In the solutions obtained in this paper, some develop a singularity at a finite point, i.e. for a fixed $t = 0$, there always exists $x = x_0$ at which the solutions blow up. There is much interest in the so-called hot spots or blow up of solutions [16, 17]. It seems that these singular solutions will model this physical phenomena. The stability of these periodic waves is numerically studied. The results show that the linearly combined waves of cn$-$ and dn–functions can propagate stably. For the blow up solutions, the linearly combined periodic solutions of nd$-$ and sd–functions and of dc$-$ and sc–functions can also propagate stably and others can not do. The method is applicable to a large variety of nonlinear PDEs.

Acknowledgement

The author would like to express his gratitude to the referees for their valuable comments. This work has been supported by the Postdoctoral Science Foundation of China.

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